Recent Open Problems in Factorization Theory

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Outline

General Notation

General notation we will use throughout this talk:

- $\mathbb{N} := \{1, 2, 3, \dots\}$,
- $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \dots\}$,
- $\mathbb{P}$ denotes the set of primes, and
- $\mathbb{F}_q$ denotes the field of q elements.

Semigroups and Monoids

Definitions Let $(M, \cdot)$ be a pair, where M is a nonempty set and " $\cdot$ " is a binary operation on M denoted by $(b, c) \mapsto b \cdot c$ or $(b, c) \mapsto bc$.

- M is a semigroup if the operation $\cdot$ is associative: $b(cd) = (bc)d$ for all $b, c, d \in M$.
- M is a monoid if it is a semigroup and contains an identity element, that is, an element 1 such that $1 \cdot b = b \cdot 1$ for all $b \in M$.
- M is a group if it is a monoid and every element is invertible (or a unit): every element $b \in M$ has an inverse, that is, an element $c \in M$ such that $bc = cb = 1$.

Let $(M, \cdot)$ be a semigroup, and write (from now on) M instead of $(M, \cdot)$.

- M is commutative if $bc = cb$ for all $b, c \in M$.
- M is cancellative if $bc = bd$ (or $cb = db$) implies $c = d$ for all $b, c, d \in M$.
- M is torsion-free if $b^n = c^n$ implies $b = c$ for all $b, c \in M$ and $n \in \mathbb{N}$.

Remark: Every monoid in this presentation is assumed to be cancellative and commutative!

Two Related Groups

Definitions Let M be a monoid.

- $M^\times$ denote the set of invertible elements (or units) of M.
- $M^\times$ is a group, called the group of units of M.
- M is called reduced if $M^\times$ is the trivial group, that is, the group containing only one element.
- M can be minimally embedded into an abelian group $\text{gp}(M)$, which is often called the Grothendieck group of M.

Example: Consider the multiplicative monoid $\mathbb{N}$.

- $\mathbb{N}^\times = \{1\}$;
- $\text{gp}(M) = \mathbb{Q}_{>0}$.

Example: Now consider the (additive) monoid $M := \mathbb{Z} \times \mathbb{N}_0$.

- $M^\times = \mathbb{Z} \times \{0\}$;
- $\text{gp}(M) = \mathbb{Z} \times \mathbb{Z}$.

Atomicity

Definition: Let M be a monoid.

- $a \in M \setminus M^\times$ is an atom (or an irreducible) if for any $b, c \in M$ the equality $a = bc$ implies that either $b \in M^\times$ or $c \in M^\times$.
- We let $\mathcal{A}(M)$ denote the set of atoms of M.
- An element of M is atomic if it is invertible or it factors into atoms.

Definition

A monoid M is atomic if every element of M is atomic.

Example: Consider the multiplicative monoid $\mathbb{N}$.

- $\mathcal{A}(\mathbb{N}) = \mathbb{P}$ and so $\mathbb{N}$ is atomic by the FTA.

Example: Now consider the (additive) monoid $M := \mathbb{Z} \times \mathbb{N}_0$.

- $\mathcal{A}(M) = \mathbb{Z} \times \{1\}$ (and $M^\times = \mathbb{Z} \times \{0\}$), and so M is atomic.

Example: $(\mathbb{R}_{\geq 0}, +)$ has empty set of atoms, and so it is not atomic.

The ACCP

Let M be a monoid.

Notation: For any $S, T \subseteq M$ and $b \in M$,

- $S \cdot T := \{st : s \in S \text{ and } t \in T\}$, and
- $b \cdot T := \{bt : t \in T\}$.

Definitions:

- A subset I of M is an ideal if $I \cdot M \subseteq I$.
- An ideal of the form $b \cdot M$ for some $b \in M$ is called principal.
- A sequence $(S_n)_{n \in \mathbb{N}}$ of subsets of M
  - is ascending if $S_n \subseteq S_{n+1}$ for every $n \in \mathbb{N}$,
  - and stabilizes if there exists $N \in \mathbb{N}$ such that $S_n = S_N$ for every $n \geq N$.
- M satisfies the ascending chain condition for principal ideals (ACCP) if every ascending chain of principal ideals stabilizes.

Proposition (well known)

Every monoid satisfying the ACCP is atomic.

An Atomic Monoid not Satisfying the ACCP

Definitions: Let M be a monoid and S a subset of M.

- A submonoid of M is a subset of M that contains 1 and is closed under the operation of M.
- The smallest submonoid of M containing S is denoted by $\langle S \rangle$ and is called the submonoid of M generated by S.
- A submonoid of $(\mathbb{Q}_{\ge 0}, +)$ is called a Puiseux monoid.

Example: Let p_n denote the n-th odd prime, and consider the Puiseux monoid

$$G := \Big\langle \frac{1}{2^n p_{n+1}} \mid n \in \mathbb{N}_0 \Big\rangle.$$

1. $\mathcal{A}(G) = \big\{ \frac{1}{2^n p_{n+1}} \mid n \in \mathbb{N}_0 \big\}$, and so G is atomic.
2. $\big(\frac{1}{2^n} + G \big)_{n \ge 1}$ is an ascending chain of principal ideals of G that does not stabilize; thus, G does not satisfy the ACCP.
3. G is called the Grams' monoid.

Integral Domains

Definitions: Let $(R, +, \cdot)$ be a triple, where R is a nonempty set and "$+$" and "$\cdot$" are two binary operations on R.

- R is a commutative ring if $(R, +)$ is a monoid, $(R, \cdot)$ is a commutative semigroup, and $b(c+d) = bc + bd$ for all $b, c, d \in R$.
- R is an integral domain if R is a commutative ring and $(R \setminus \{0\}, \cdot)$ is a monoid.
- R is a field if R is an integral domain and $(R \setminus \{0\}, \cdot)$ is a group.

Let R be an integral domain.

- $R^* := R \setminus \{0\}$ is called the multiplicative monoid of R,
- $R^\times := (R^*)^\times$ is called the group of units of R, and
- R is atomic (resp., satisfies the ACCP) if the monoid R^* is atomic (resp., satisfies the ACCP).

Examples:

1. $\mathbb{Z}$ is an integral domain, while $\mathbb{Q}$ and $\mathbb{R}$ are fields.
2. If $p \in \mathbb{P}$ and $n \in \mathbb{N}$, we let $\mathbb{F}_{p^n}$ denote the **only** field with p^n elements.
3. If R is a commutative ring so is the set $R[x]$ of polynomials with coefficients in R (if R is an integral domain, so is $R[x]$).

The Bounded Factorization Property

Definitions: Let M be a monoid.

- For $b \in M$, a formal product $a_1 \cdots a_\ell$ of atoms is called a factorization of b if $b = a_1 \cdots a_\ell$ in M, in which case, ℓ is called a length of b.
- The set of all possible lengths of $b \in M$ is denoted by $\mathsf{L}(b)$.
- M is a bounded factorization monoid (BFM) if $\mathsf{L}(b)$ is nonempty and finite for every $b \in M$.
- An integral domain R is a bounded factorization domain (BFD) if R^* is a BFM.

Examples:

- $\mathbb{Z}$ is a BFD.
- If R is a BFD, then $R[x]$ is a BFD.
- Thus, $\mathbb{Z}[x]$ is a BFD.

Proposition (well known)

Every BFM/BFD satisfies the ACCP.

An ACCP Monoid that is not a BFM

Example: Let p_n denote the n-th prime, and consider the Puiseux monoid

$$M := \Big\langle \frac{1}{p_n} \mid n \in \mathbb{N} \Big\rangle.$$

1. $\mathcal{A}(M) = \big\{ \frac{1}{p_n} \mid n \in \mathbb{N} \big\}$, and so M is atomic.
2. For every $b \in M$, there exists unique $n_b \in \mathbb{N}_0$ and $c_n \in \{0, 1, \dots, p_n - 1\}$ such that
$$b = n_b + \sum_{n \in \mathbb{N}} c_n \frac{1}{p_n},$$
where all but finitely many c_n equal 0.
3. M satisfies the ACCP.
4. $\mathsf{L}(1) = \mathbb{P}$, and so M is not a BFM.

The Finite Factorization Property

Definitions: Let M be a monoid.

- For $b \in M$, the set of all factorizations of b is denoted by $\mathsf{Z}(b)$.
- M is a finite factorization monoid (FFM) if $\mathsf{Z}(b)$ is nonempty and finite for every $b \in M$.
- An integral domain R is a finite factorization domain (FFD) if R^* is an FFM.

Examples:

- Every field and $\mathbb{Z}$ are FFDs.
- If R is an FFD, then $R[x]$ is an FFD.
- Thus, $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are FFDs.
- Every FFM (resp., FFD) is a BFM (resp., BFD).
- $\{0\} \cup \mathbb{R}_{\ge 1}$ is a BFM that is not an FFM: $\left(\frac{3}{2} + \frac{1}{2^n}\right) + \left(\frac{3}{2} - \frac{1}{2^n}\right) \in \mathsf{Z}(3)$.

Monoid Algebras

Let R be a commutative ring, and let M be an (additive) monoid.

Definition: The monoid algebra $R[M]$ of M over R is the commutative ring consisting of all polynomial expressions with coefficients in R and exponents in M, with addition and multiplication defined as for standard polynomials.

Example: Both $x^7 + x^{3/2}$ and $2x^{3/2} + 1$ are polynomial expressions in $\mathbb{Z}[\mathbb{Q}_{\geq 0}]$. In addition,

$$\left(x^7 + x^{3/2}\right) + \left(2x^{3/2} + 1\right) = x^7 + 3x^{3/2} + 1$$

and

$$\left(x^7 + x^{3/2}\right)\left(2x^{3/2} + 1\right) = 2x^{7+3/2} + x^7 + 2x^3 + x^{3/2}.$$

Proposition (well known)

$R[M]$ is an integral domain if and only if R is an integral domain and M is torsion-free.

Proposition (well known)

Let F be a field, and assume that M is a torsion-free monoid.

- *$F[M]$ satisfies the ACCP provided that M satisfies the ACCP.*
- *$F[M]$ is a BFD provided that M is a BFM.*

Problems 1 and 2: Atomicity in Monoid Algebras

Theorem (Coykendall-G., 2019)

There exists a Puiseux monoid M such that $\mathbb{F}_2[M]$ is not atomic.

Consider the Grams' monoid: $G := \left\langle \frac{1}{2^n p_{n+1}} \mid n \in \mathbb{N}_0 \right\rangle$.

Recall: G is atomic but does NOT satisfy the ACCP.

Problem (Open Problem 1)

Is the monoid algebra $\mathbb{Q}[G]$ atomic?

For $q \in \mathbb{Q} \cap (0,1)$, consider the Puiseux monoid $M_q := \langle q^n \mid n \in \mathbb{N}_0 \rangle$.
Remark: If $q^{-1} \notin \mathbb{N}$, then M_q is atomic with $\mathcal{A}(M_q) = \{q^n \mid n \in \mathbb{N}_0\}$.
However, M_q does NOT satisfy the ACCP.

Problem (Open Problem 2)

Is the monoid algebra $\mathbb{Q}[M_q]$ atomic?

Problems 3 and 4: Factorizations in Monoid Algebras

Recall:

- An atomic monoid M is an FFM if every element of M has only finitely many factorizations.
- An integral domain R is an FFD if its multiplicative monoid R^* is an FFM.

Problem (Open Problem 3)

Is there an FFM Puiseux monoid M such that $\mathbb{Q}[M]$ is not an FFD?

Definitions:

- An atomic monoid M is an LFFM if every element of M has only finitely many factorizations of any prescribed length.
- An integral domain R is an LFFD if its multiplicative monoid R^* is an LFFM.

Remark: Every FFM (resp., FFD) is an LFFM (resp., LFFD).

Problem (Open Problem 4)

Is there an LFFM Puiseux monoid M such that $\mathbb{Q}[M]$ is not an LFFD?

Problems 5: Hereditary Atomicity VS ACCP

Definitions: A monoid M is called hereditarily atomic if every submonoid of M is atomic.

Theorem (G.-Li, 2023)

Every hereditarily atomic monoid satisfies the ACCP.

Let R be a commutative ring.

- A subset of R is called a subring if it contains 1 and it is a a commutative ring with the operations inherited from R.
- An integral domain R is called hereditarily atomic if every subring of R is atomic.

Remark: The integral domain $\mathbb{Q}[x]$ clearly satisfies the ACCP, but contains the non-atomic subring $\mathbb{Z} + x\mathbb{Q}[x]$. Thus, $\mathbb{Q}[x]$ is not hereditarily atomic.

Problem (Open Problem 5)

Does every hereditarily atomic integral domain satisfy the ACCP?

Problems 6: Overatomicity VS ACCP

Definitions: Let M be a monoid.

- An overmonoid of M is a submonoid M' of $\text{gp}(M)$ such that $M \subseteq M' \subseteq \text{gp}(M)$.
- M is called overatomic if every overmonoid of M is atomic.

Observation: Every overatomic monoid is atomic.

Problem (Open Problem 6)

Does every overatomic monoid satisfy the ACCP?

Example: Consider the monoid $M := \{(0,0)\} \cup (\mathbb{Z} \times \mathbb{N})$.

1. M is reduced and $\mathcal{A}(M) = \{(n,1) \mid n \in \mathbb{Z}\}$. Thus, M is atomic.
2. Moreover, the only length of $(m,n) \in M$ is n, and so M is a BFM.
3. $M' := (\mathbb{N}_0 \times \{0\}) \cup (\mathbb{Z} \times \mathbb{N})$ is an overmonoid of M with $\mathcal{A}(M') = \{(1,0)\}$. Therefore M' is not atomic.
4. Hence M satisfies the ACCP but is not overatomic.

Subatomicity

Definition: Let M be a monoid.

- A monoid M is quasi-atomic if for every non-invertible element $c \in M$, there exists an element $b \in M$ such that bc is atomic.
- A monoid M is almost atomic if for every non-invertible element $c \in M$, there exists an atomic element $b \in M$ such that bc is atomic.
- An integral domain is quasi-atomic (resp., almost atomic) if its multiplicative monoid is quasi-atomic (resp., almost atomic).

Observations:

- Every atomic monoid/domain is almost atomic.
- Every almost atomic monoid/domain is quasi-atomic.

Remark: None of the previous two implications is reversible (Examples have been constructed by F. G., N. Lebowitz-Lockard, C. Liu, P. Rodriguez, M. Tirador, J. Vulakh)

Problems 7 and 8: Subatomicity in Polynomial Rings

Theorem (Roitman, 1993)

There exists an atomic integral domain R such that $R[x]$ is not atomic.

In the same direction of Roitman's theorem, one can try to answer the following questions.

Problem (Open Problem 7)

Is there a quasi-atomic domain R such that $R[x]$ is not quasi-atomic?

Problem (Open Problem 8)

Is there an almost atomic domain R such that $R[x]$ is not almost atomic?

Problems 9 and 10: Subatomicity in Monoid Algebras

Problem (Open Problem 9)

Is there a quasi-atomic Puiseux monoid M such that $\mathbb{Q}[M]$ is not quasi-atomic?

Problem (Open Problem 10)

Is there an almost atomic Puiseux monoid M such that $\mathbb{Q}[M]$ is not almost atomic?

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End of Presentation

THANK YOU!